WEAK GALERKIN METHOD FOR LINEAR ELASTICITY

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ABSTRACT

We propose a weak Galerkin (WG) method for linear elasticity based on the primal formulation. We also introduce a simple post-processing technique to obtain a numerical approximation of the stress. A-priori error estimates of optimal order for the displacement and stress are proved and several numerical experiments confirm the theoretical results.

INTRODUCTION

Consider an isotropic elastic body in the configuration space \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), and assume that \( \Omega \) is an open bounded and connected domain with Lipschitz continuous boundary \( \partial \Omega \). The kinematic model for linear elasticity then reads as follows. Given an exterior body force \( f \), find the displacement \( u \in \mathbb{R}^d \) that satisfies

\[
- \nabla \cdot (2\mu \varepsilon(u) + \lambda (\nabla \cdot u)I) = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g_D \quad \text{on} \quad \partial \Omega.
\]

(1)

Here, \( \varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) \) is the linear strain tensor, \( \mu \) and \( \lambda \) are Lamé constants, \( I \) is a \( d \times d \) identity tensor. Also, the stress tensor is defined by

\[
\sigma = 2\mu \varepsilon(u) + \lambda (\nabla \cdot u)I.
\]

(2)

Rewriting the equation (1) by using \( \nabla \cdot (\nabla u)^T = \nabla \cdot ((\nabla \cdot u)I) \), then multiplying the resulting equation by a test function \( v \in (H_0^1(\Omega))^d \) and using integration by parts, we arrive at the weak formulation

\[
\mu (\nabla u, \nabla v) + (\mu + \lambda) (\nabla \cdot u, \nabla \cdot v) = (f, v) \quad \forall v \in (H_0^1(\Omega))^d.
\]

(3)

It is well known that standard continuous Galerkin methods, such as continuous piecewise linear or bilinear elements, yield poor approximations to the displacement as the material becomes incompressible or equivalently, the Lamé constant \( \lambda \to \infty \). This phenomenon is known as “Poisson locking” and overcoming locking has been the subject of extensive research over several decades. The main goal of this study is to develop and mathematically analyze a WG method that is locking-free and computationally efficient.
A LOWEST-ORDER WEAK GALERKIN METHOD [1]

Let \( T_h \) denote a shape regular triangulation of \( \Omega \) into simplicial or rectangular meshes and let \( E_h \) be the union of all edges/faces of the mesh elements. For each \( T \in T_h \), \( h_T \) denotes the diameter of \( T \) and the mesh size of \( T_h \) is defined as \( h = \max_{T \in T_h} h_T \).

Our global weak finite element space \( V_h \) is
\[
V_h = \{ v = \{ v_0, v_b \} : v_0|_T \in [P_0(T)]^d \ \forall T \in T_h, \ v_b|_e \in [P_0(e)]^d \ \forall e \in E_h \}.
\]

In WG methods, classical differential operators are replaced by discrete weak differential operators in the variational formulation of the underlying PDE problem. We define our weak divergence and weak gradient operators as follows: For any \( v \in V_h \), the local discrete weak divergence \( \nabla_{w,T} \cdot v \in P_0(T) \) and the discrete weak gradient \( \nabla_{w,T} v \in (RT_0(T))^d \) satisfy, respectively,
\[
(\nabla_{w,T} \cdot v, \phi)_T = -(v_0, \nabla \phi)_T + (v_b \cdot n, \phi)_{\partial T} \quad \forall \phi \in P_0(T)
\]
and
\[
(\nabla_{w,T} v, \psi)_T = -(v_0, \nabla \psi)_T + (v_b, \psi n)_{\partial T} \quad \forall \psi \in (RT_0(T))^d.
\]
Here, \( n \) is the unit outward normal vector on \( \partial T \) and \( RT_0 \) is the lowest-order Raviart-Thomas space.

Then, the weak Galerkin method to solve (1) is to find \( u_h = \{ u_{h0}, u_{hb} \} \in V_h \) with \( u_{hb} = Q_b g_D \) on \( \partial \Omega \) such that
\[
\mu \sum_{T \in T_h} (\nabla_{w} u_h, \nabla_{w} v)_T + (\mu + \lambda) \sum_{T \in T_h} (\nabla_{w} \cdot u_h, \nabla_{w} \cdot v)_T = (f, v_0) \quad \forall v = \{ v_0, v_b \} \in V_h^0,
\]
where \( Q_b \) is the local \( L^2 \)-projection onto a set of all piecewise constant vector functions on \( E_h \).

After finding the approximate displacement \( u_h \), we post-process it to approximate the stress by computing
\[
\sigma_h = 2\mu \varepsilon_{w}(u_h) + \lambda(\nabla_{w} \cdot u_h)I,
\]
where \( \varepsilon_{w}(u_h) = \frac{1}{2}(\nabla_{w} u_h + (\nabla_{w} u_h)^T) \).

A-PRIORI ERROR ESTIMATES

Theorem 0.1 Let \( u \) and \( u_h \) be the solution of (3) and (4), respectively, and let \( \sigma \) and \( \sigma_h \) be the stress tensor and the approximate stress tensor defined in (2) and (5), respectively. Then, we have the following error estimates:
\[
\| \nabla u - \nabla_{w} u_h \|_0 \leq Ch\|f\|_0, \\
\| u - u_{h0} \|_0 \leq Ch\|f\|_0, \\
\| \sigma - \sigma_h \|_0 \leq Ch\|f\|_0,
\]
where \( C \) is independent of \( h \) and \( \lambda \).

REFERENCES