On Boundary Layers for the Burgers Equations in a Bounded Domain

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ABSTRACT

As a simplified model derived from the Navier-Stokes equations, we consider the viscous Burgers equations in a bounded domain with two-point boundary conditions.

\[
\begin{align*}
    u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} + \frac{(u^\varepsilon)^2}{2} &= f(x, t), \quad x \in (0, 1), \quad t \geq 0 \\
    u^\varepsilon(x, 0) &= u_0(x), \quad x \in (0, 1), \\
    u^\varepsilon(0, t) &= g(t), \quad t \geq 0, \\
    u^\varepsilon(1, t) &= h(t), \quad t \geq 0.
\end{align*}
\]

(1)

We investigate the singular behaviors of their solutions \(u^\varepsilon\) as the viscosity parameter \(\varepsilon\) gets smaller. Indeed, when \(\varepsilon\) gets smaller, \(u^\varepsilon_x\) has \(1/\varepsilon\) order slope. So controlling the sharp layer is one of the most important parts in this research.

The idea is constructing the asymptotic expansions in the order of the \(\varepsilon\) and validating the convergence of the expansions to the solutions \(u^\varepsilon\) as \(\varepsilon \to 0\) in \(L^2(0, T; H^1((0, 1)))\) space. In this article, we consider the case where sharp transitions occur at the boundaries, i.e. boundary layers, and we fully analyse the convergence at any order of \(\varepsilon\) using the so-called boundary layer correctors as follows.

**Theorem 1** Let \(u^\varepsilon\) be the solutions (1) and let \(u^j\) and \(\tilde{\theta}^j\left(\frac{x}{\varepsilon}, t\right)\) be the solutions of proper equations for \(j = 0, \cdots, n\). Assume further that

\[
\begin{align*}
    u^n &\in L^\infty(0, T; H^2(\Omega)), \quad u^n_\varepsilon \in L^\infty(0, T; H^1(\Omega)), \\
    \text{and for } j = 0, 1, \cdots, n-1, \\
    w^j &\in L^\infty(0, T; H^2(\Omega)), \quad w^j_\varepsilon \in L^\infty(0, T; W^{1,\infty}(\Omega)).
\end{align*}
\]

(2)

(3)

Then there exists a constant \(C > 0\), independent of \(\varepsilon\), such that

\[
\sup_{0 \leq t \leq T} \left\| u^\varepsilon - \sum_{j=0}^n \varepsilon^j \left(u^j + \tilde{\theta}^j\right) \right\|_{L^2(\Omega)} + \varepsilon^{\frac{n}{2}} \left\| u^\varepsilon - \sum_{j=0}^n \varepsilon^j \left(u^j + \tilde{\theta}^j\right) \right\|_{L^2(0, T; H^1(\Omega))} \leq C\varepsilon^{n+1}.
\]

(4)

In the end, we also numerically verify the convergences.
REFERENCES


