Improving accuracy of the fifth-order WENO scheme by using the exponential approximation space

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ABSTRACT

The aim of this study is to develop a novel WENO scheme that improves the performance of the well-known fifth-order WENO methods. The approximation space consists of exponential polynomials with a tension parameter that may be optimized to fit the the specific feature of the data, yielding better results compared to the polynomial approximation space. However, finding an optimal tension parameter is a very important and difficult problem, indeed a topic of active research. In this regard, this study introduces a practical approach to determine an optimal tension parameter by taking into account the relationship between the tension parameter and the accuracy of the exponential polynomial interpolation under the setting of the fifth-order WENO scheme. As a result, the proposed WENO scheme attains an improved order of accuracy (that is, sixth-order) better than other fifth-order WENO methods without loss of accuracy at critical points. A detailed analysis is provided to verify the improved convergence rate. Further, we present modified nonlinear weights based on L^1-norm approach along with a new global smoothness indicator. The proposed nonlinear weights reduce numerical dissipation significantly, while attaining better resolution in smooth regions. Some experimental results for various benchmark test problems are presented to demonstrate the ability of the new scheme.

INTRODUCTION

Hyperbolic systems are used for a wide range of scientific and engineering applications such as meteorology, gas dynamics, shallow water modeling, astrophysics models, and multiphase flow problems.

The space of algebraic polynomials is the most well-established tool to reconstruct numerical flux. However, the interpolation method cannot be regulated according to the trait of the given data such that it causes excessive numerical dissipation when approximating rapidly varying data (e.g., sharp gradients or high oscillations). To circumvent this limitation, this study exploits the interpolation method based on the space of exponential polynomials of the form

$$\phi(x) = x^k e^{\lambda x}, \quad k \in \mathbb{Z}_+, \lambda \in \mathbb{R} \cup i\mathbb{R} \tag{1}$$

that allows an environment to fit the approximation to the characteristic of the given problem. For a given exponential polynomial space, the choice of the tension (or shape) parameter λ has a significant impact on the accuracy of interpolation. A well-selected parameter can yield better results compared to the polynomial-based method for various types of PDEs [1,2,3,4,5].

However, selecting an optimal parameter is an important and difficult problem, indeed a topic of active research. Most studies end up finding the tension parameter by using trial and error or minimization problem. In this regards, the goal of this study is first to present a specific type of exponential approximation space for the construction of numerical fluxes under the setting of the fifth-order WENO scheme. We then introduce a practical approach to determine an optimal parameter by taking into account the relation between the value of the tension parameter and the accuracy of the exponential polynomial interpolation. As a result, the proposed WENO scheme (termed as WENO-H) provides an improved order of accuracy better than the other fifth-order WENO methods. In fact, we will observe that the sixth-order accuracy can be achieved by the WENO-H technique, without loss of accuracy at critical points. A rigorous analysis is provided to prove the improved convergence rate.

Further, a modified smoothness indicator based on L^1-norm approach is presented along with a new global smoothness indicator. Accordingly, the proposed WENO scheme reduces numerical dissipation significantly, while attaining better resolution in smooth regions. Some experimental results for various benchmark test problems are given to illustrate the performance of the WENO-H scheme.

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