ADAPTIVE MULTI-LEVEL ALGORITHM FOR NONLINEAR PROBLEMS

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ABSTRACT

In this paper, we derive a unified a posteriori error estimate for the approximate solutions of the multi-level algorithm developed in D. Kim, E.-J. Park & B. Seo [1] for a class of nonlinear problems. The multi-level algorithm using a two-grid idea is an efficient numerical method designed to resolve nonlinearities. The two-grid algorithm is first applied and in the subsequent process, uniform mesh refinement is exploited to deliver a desired accuracy and convergence; as the mesh is being refined, the solution on a given mesh is exploited as an accurate starting iterate for solutions on the next mesh level. The multi-level algorithm has quadratic convergence in the sense of (3). We apply this strategy to efficiently compute numerical solutions by adaptive mesh refinement. We emphasize that the multi-level algorithm on the adaptive meshes retains quadratic convergence of Newton's method across different mesh levels, which is validated from the numerical results presented in the last section. In particular, existing a posteriori error estimates for the linear problem can be utilized to find reliable error estimators for the nonlinear problem. We mention that this approach is quite general and can be applied to any numerical scheme for a class of nonlinear problems.

As applications, we consider the pseudostress-velocity formulation of the stationary Navier-Stokes equations (NSE) and the standard Galerkin formulation of a semilinear elliptic equations (SEE). The mixed finite element methods for the pseudostress-velocity formualtion have been recently established in for the Stokes [2] & [3] and the Navier-Stokes equations [4] & [5]. The pseudostress-velocity formualtion has several advantages: The pseudostress is nonsymmetric unlike stress tensor and approximated by the RT_k elements, Raviart-Thomas elements of index $k \geqslant 0$. Moreover, the approximation of the pressure, the velocity gradient, or even the stress can be algebraically obtained from the approximate value of the pseudostress.

MULTI-LEVEL ALGORITHM

For given $\mathcal{C}^2$ map $G : \Lambda \times X \to Y$, we consider the following nonlinear problem: Find $(\nu, \phi) \in \Lambda \times X$ such that

$$F(\nu, \phi) := \phi + SG(\nu, \phi) = 0, \tag{1}$$

where $S \in \mathscr{L}(Y; \mathscr{X})$ is a linear operator independent of ν.

The approximation of nonlinear problem (1) is to find $\phi^h \in X$ such that

$$F_h(\nu, \phi^h) := \phi^h + S_h G(\nu, \phi^h) = 0. \tag{2}$$

Algorithm: Multi-level Algorithm
Step 1: (Nonlinear Solver) Solve nonlinear system on initial mesh
Find $\phi_0(\nu) \in X_0$ such that $F_0(\nu, \phi_0(\nu)) = 0$.
Step 2: (Linear Solver) Update on each mesh level j with one Newton iteration
For $j = 1, 2, \cdots$, find $\phi_j(\nu) \in X_j$ such that $D_\phi F_j(\nu, \phi_{j-1}(\nu))(\phi_j(\nu) - \phi_{j-1}(\nu)) = -F_j(\nu, \phi_{j-1}(\nu))$.

Theorem 1 ***(a priori error bound)*** *Assume that (H1)-(H4) hold. Let $\phi^{h_j}(\nu)$ be nonsingular solutions of (2) on X_j. And let $\phi_j(\nu)$ be solutions obtained from **Step 2** of **Algorithm**. Then there exist $\xi > 0$ with $\xi \leqslant \bar{\xi}$ and $h_0^* > 0$ such that for $h_j \leqslant h_0^*$, $\phi^{h_j}(\nu) \in B(\phi(\nu), \xi/2)$ and $\phi_j(\nu)$ belongs to the ball $B(\phi^{h_j}(\nu), \xi/2)$ for $j \geqslant 1$. Moreover, for positive constants K_2 & K_3 independent of mesh sizes h_j and ν, we have the following quadratic relation*

$$\|\phi^{h_j}(\nu) - \phi_j(\nu)\|_X \leqslant K_2 \|\phi^{h_j}(\nu) - \phi_{j-1}(\nu)\|_X^2, \tag{3}$$

and we have an a priori estimate

$$\begin{aligned}\|\phi(\nu) - \phi_j(\nu)\|_X \leqslant & K_3 \Big(\|(S - S_j)G(\nu, \phi(\nu))\|_{\mathscr{X}} + \|(S - S_j)G(\nu, \phi(\nu))\|_{\mathscr{X}}^2 \\ & + \|\phi(\nu) - \phi_{j-1}(\nu)\|_X^2 \Big).\end{aligned}$$

Theorem 2 ***(a posteriori error bound)*** *Assume that (H1)-(H4) hold. Let $\phi(\nu)$ be a nonsingular solution of (1) and let $\phi_j(\nu)$ be the approximate solution produced by **Algorithm**. Then, there exists $0 < h_1^* \leqslant h_0^*$ given in Theorem 1 such that for all $h_j \leqslant h_1^*$, we have $\phi^{h_j}(\nu) \in B(\phi(\nu), \xi/2)$ and $\phi_j(\nu) \in B(\phi^{h_j}(\nu), \xi/2)$ for ξ given in Theorem 1, and we have a residual type a posteriori error bound with a positive constant K_1 independent of mesh sizes h_j and ν*

$$\|\phi(\nu) - \phi_j(\nu)\|_X \leqslant 2\|D_\phi F(\nu, \phi_j(\nu))^{-1}\|_{\mathscr{X};X} \|F(\nu, \phi_j(\nu))\|_{\mathscr{X}} \leqslant K_1 \|F(\nu, \phi_j(\nu))\|_{\mathscr{X}}.$$

REFERENCES

1. D. Kim, E.-J. Park, and B. Seo, *Convergence of Multi-level Algorithms for a Class of Nonlinear Problems*, J. Sci. Comput. 84 (2020), no. 2, 34.
2. Z. Cai and Y. Wang, *A multigrid method for the pseudostress formulation of Stokes problems*, SIAM J. Sci. Comput., 29 (2007), No. 5, pp. 2078–2095.
3. C. Carstensen, D, Kim, and E. J. Park, *A priori and a posteriori pseudostress-velocity mixed finite element error analysis for the Stokes problem*, SIAM J. Numer. Anal., 49 (2011), No. 6, pp. 2501–2523.
4. Z. Cai, C. Wang, and S. Zhang, *Mixed finite element methods for incompressible flow: stationary Navier-Stokes equations*, SIAM J. Numer. Anal., 48 (2010), No. 1, pp. 79–94.
5. D. Kim, E.-J. Park, and B. Seo, *Optimal error estimates for the pseudostress formulation of the Navier-Stokes equations*, Applied Mathematics Letters, 78 (2018), pp. 24–30