TAIL ASYMPTOTIC OF A RANDOM SUM OF A LONG-TAILED NUMBER OF RANDOM VARIABLES

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ABSTRACT

We study the tail asymptotic of a random sum $S_N = Z_1 + \cdots + Z_N$ of independent and identically distributed real-valued random variables Z_i where N is a nonnegative integer-valued random variable with a regularly varying distribution, independent of Z_i. Under certain conditions about the moment and the tail distribution of Z_1, we obtain the tail asymptotic of S_N.

LONG-TAILED DISTRIBUTION

A distribution F is said to be heavy-tailed if

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) = \infty \text{ for all } \lambda > 0,$$

or equivalently [1]

$$\limsup_{x\to\infty} \overline{F}(x) e^{\lambda x} = \infty \text{ for all } \lambda > 0$$

where $\overline{F} := 1 - F$. A distribution F is said to be light-tailed if F is not heavy-tailed.

Heavy-tailed distributions play a major role in the analysis of many stochastic systems. For instance, they are also frequently used to accurately model inputs to computer and communications networks.

The class of long-tailed distributions is one of important subclasses of heavy-tailed distributions [1].

Long-tailed Distribution

A distribution function F is said to be long-tailed if

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \text{ for all } y > 0,$$

or equivalently

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \text{ for some } y \neq 0.$$

Regularly Varying Distribution

A distribution function F is said to be regularly varying at infinity with index $-\alpha < 0$ if,

$$\text{for any fixed } c > 0, \lim_{x\to\infty} \frac{\overline{F}(cx)}{\overline{F}(x)} = c^{-\alpha} \text{ as } x \to \infty.$$

The class of long-tailed distributions contains the class of regularly varying distributions [1]. The class of regularly varying distributions is important from the perspective of 'Maximal domain of attraction' [3]. A few of well-known heavy-tailed distributions, e.g., the Pareto, Burr, and Cauchy distributions, are regularly varying at infinity [1]. If a distribution F on $\mathbb{R}$ is regularly varying at infinity with index $-\alpha < 0$, then all positive moments of order $\gamma < \alpha$ are finite, while all positive moments of order $\gamma > \alpha$ are infinite [1,2] .

MAIN RESULTS

Based on the properties of long-tailed distributions, we obtain three main results:

(1) Let N be a nonnegative integer-valued random variable and Z_i, $i = 1, 2, ...$, be a sequence of i.i.d. real-valued random variables such that $|Z_1|$ has a light-tailed distribution and $-\infty < \mathbb{E}[Z_1] := a < 0$. Then $\sum_{i=1}^{N} Z_i$ has a light-tailed distribution.

(2) Let N be a nonnegative integer-valued random variable with a regularly varying distribution with index $-\alpha < 0$ and Z_i, $i = 1, 2, ...$, be a sequence of i.i.d. real-valued random variables, independent of N, such that $|Z_1|$ has a light-tailed distribution and $0 < \mathbb{E}[Z_1] := a < \infty$. Then $\Pr\left\{\sum_{i=1}^{N} Z_i > x\right\} \sim \Pr\{aN > x\}$ as $x \to \infty$.

(3) Let N be a nonnegative integer-valued random variable with a regularly varying distribution with index $-\alpha < 0$ satisfying $\mathbb{E}[N] < \infty$ and Z_i, $i = 1, 2, ...$, be a sequence of i.i.d. real-valued random variables, independent of N, such that $0 < \mathbb{E}[Z_i] := a < \infty$, $\mathbb{E}[(Z_i^+)^r] < \infty$ for some $r > 1$ and $\Pr\{Z_i > x\} = o(\Pr\{N > x\})$ as $x \to \infty$. Then $\Pr\left\{\sum_{i=1}^{N} Z_i > x\right\} \sim \Pr\{aN > x\}$ as $x \to \infty$.

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