Estimates of anisotropic Sobolev spaces with mixed norms for the Stokes system in a half-space

Tongkeun Chang [1] and Kyungkeun Kang [2]

1) *Department of Mathematics, Yonsei University, Seoul 120-749, South Korea*

2) *Department of Mathematics, Yonsei University, Seoul 120-749, South Korea*

ABSTRACT

We are concerned with the non-stationary Stokes system with non-homogeneous external force and non-zero initial data in $\mathbb{R}^n_+ \times (0,T)$. We obtain new estimates of solutions including pressure in terms of mixed anisotropic Sobolev spaces. As an application, some anisotropic Sobolev estimates are presented for weak solutions of the Navier-Stokes equations in a half-space in dimension three. To be more precise, we study the non-stationary Stokes system in a half-space $\mathbb{R}^n_+ \times [0,T)$, $n \geq 3$

$$v_t - \Delta v + \nabla p = f, \qquad \operatorname{div} v = 0 \qquad \text{in } Q^+_T := \mathbb{R}^n_+ \times [0,T), \tag{1}$$

where $v : Q^+_T \to \mathbb{R}^n$ is the velocity field and $p : Q^+_T \to \mathbb{R}$ is the pressure. We consider the initial and boundary value problem of (1), whereby no slip boundary conditions are assigned, that is

$$v(x,0) = v_0(x) \quad \text{and} \quad v(x,t) = 0, \quad x \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}. \tag{2}$$

One of main results reads as follows:

Theorem 0.1 *Let $0 \leq \alpha \leq 2$, $0 < T \leq \infty$ and $1 < p, q < \infty$. Suppose that $f \in H^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))$. and $v_0 \in B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ with $\operatorname{div} f = 0$ and $\operatorname{div} v_0 = 0$ in the sense of distributions. Then, there exists a unique solution v in $H^{\alpha,\frac{1}{2}\alpha}_{pq}(\mathbb{R}^n_+ \times (0,T))$ of (1) such that the following estimate is satisfied:*

$$\|v\|_{H^{\alpha,\frac{1}{2}\alpha}_{pq}(\mathbb{R}^n_+ \times (0,T))} \leq c\|f\|_{H^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))} + c\|v_0\|_{B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}. \tag{3}$$

Moreover, if $1 + 1/p < \alpha \leq 2$, then $p \in L_q(0,T; H^{\alpha-1}_p(\mathbb{R}^n_+))$ such that

$$\|p\|_{L_q(0,T;H^{\alpha-1}_p(\mathbb{R}^n_+))} \leq c\|f\|_{H^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))} + c\|v_0\|_{B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}. \tag{4}$$

If $f \in \dot{H}^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))$ and $v_0 \in \dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$, then $v \in \dot{H}^{\alpha,\frac{1}{2}\alpha}_{pq}(\mathbb{R}^n_+ \times (0,T))$ and satisfies

$$\|v\|_{\dot{H}^{\alpha,\frac{1}{2}\alpha}_{pq}(\mathbb{R}^n_+ \times (0,T))} \leq c\|f\|_{\dot{H}^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))} + c\|v_0\|_{\dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}. \tag{5}$$

Furthermore, if $1 + 1/p < \alpha \leq 2$, then $p \in L_q(0,T; \dot{H}^{\alpha-1}_p(\mathbb{R}^n_+))$ such that

$$\|p\|_{L_q(0,T;\dot{H}^{\alpha-1}_p(\mathbb{R}^n_+))} \leq c\|f\|_{\dot{H}^{\alpha-2,\frac{1}{2}\alpha-1}_{pq,0}(\mathbb{R}^n_+ \times (0,T))} + c\|v_0\|_{\dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}. \tag{6}$$