Mathematical Analysis on the Inverse Problem of Quantitative Susceptibility Mapping

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ABSTRACT

Quantitative susceptibility mapping (QSM) provides a novel contrast mechanism in Magnetic Resonance Imaging (MRI) different from traditional Susceptibility Weighted Imaging (SWI) and it is expected to be effective in diagnosing diseases in a brain such as Alzheimer's disease and Parkinson's disease. The inverse problem of QSM is to recover the susceptibility distribution of the human body from the measured local field that is expressed by the convolution of the susceptibility distribution with the magnetic field generated by a unit dipole. More precisely, the inverse problem of QSM aims to reconstruct the susceptibility χ from the given relative difference field ψ via the relation:

$$\psi(\boldsymbol{x}) = d * \chi(\boldsymbol{x}) = \int_{\mathbb{R}^3} d(\boldsymbol{x}-\boldsymbol{y})\chi(\boldsymbol{y})d\boldsymbol{y} \tag{1}$$

where $d(\boldsymbol{x})$ is the dipole kernel defined as

$$d(\boldsymbol{x}) = \frac{2x_3^2 - x_1^2 - x_2^2}{4\pi|\boldsymbol{x}|^5}.$$

Alternatively, one can try to recover χ from ψ via the relation in Fourier domain:

$$\Psi(\boldsymbol{\xi}) = \mathcal{D}(\boldsymbol{\xi})\mathcal{X}(\boldsymbol{\xi}) = \left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)\mathcal{X}(\boldsymbol{\xi}) \tag{2}$$

However, (2) is ill-posed due to the existence of the zero cone $\Gamma_0 = \Big\{\boldsymbol{\xi} \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 - 2\xi_3^2 = 0\Big\}$, and due to this ill-posedness, the reconstructed image contains severe streaking artifacts. To overcome these streaking artifacts, numerous reconstruction methods have been developed to pursue the scientific and/or clinical application. However, rigorous mathematical analyses for the inverse problem, such as demonstrations of the existence and uniqueness of solutions and error characterizations, have not yet been presented. In this presentation, we will provide such rigorous mathematical analyses for the inverse problem of QSM.

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