Adaptive Crank-Nicolson Mixed Finite Elements for Parabolic Problem

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요 약

In this paper we present an a posteriori error estimator for fully discrete solutions of linear parabolic problems. The discretization is used lowest order Raviart-Thomas finite elements in space and Crank-Nicolson in time. Adjusting Crank-Nicolson reconstruction idea introduced by Akrivis, Makridakis & Nochetto [1], we construct an a posteriori error estimator of second order in time for the Crank-Nicolson mixed FEM.

INTRODUCTION

We consider the following linear parabolic equation

$$\begin{aligned} u_t - \triangle u = f \quad & \text{in } \Omega \times (0, \mathrm{T}), \\ u = 0 \quad & \text{on } \partial\Omega \times (0, \mathrm{T}), \\ u(\cdot, 0) = u_0 \quad & \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain; $u = u(x,t)$, u_t denotes $\partial u/\partial t$. The problem (1) can be written as the following first-order system: *find* $(u, \boldsymbol{\sigma})$ *such that*

$$\begin{aligned} \boldsymbol{\sigma} := -\nabla u, \ u_t + \mathrm{div}\boldsymbol{\sigma} = f \quad & \text{in } \Omega \times (0, \mathrm{T}), \\ u = 0 \quad & \text{on } \partial\Omega \times (0, \mathrm{T}), \\ u(\cdot, 0) = u_0 \quad & \text{in } \Omega. \end{aligned} \tag{2}$$

Let $M := H^1(0, T; L^2(\Omega))$ and $\boldsymbol{X} := L^2(0, T; H(\mathrm{div}; \Omega))$. Then we have the following variational formulation of (2): *find* $(u, \boldsymbol{\sigma})$ *such that*

$$\begin{aligned} &(u(\cdot, 0), v(\cdot, 0)) + \int_0^T \Big((u_t, v) + (\mathrm{div}\boldsymbol{\sigma}, v) + (u, \mathrm{div}\boldsymbol{q}) - (\boldsymbol{\sigma}, \boldsymbol{q}) \Big) dt \\ =&(u_0, v(\cdot, 0)) + \int_0^T (f, v) dt, \quad \forall (v, \boldsymbol{q}) \in M \times \boldsymbol{X} \end{aligned} \tag{3}$$

where $(\cdot, \cdot)$ *denotes the* $L^2(\Omega)$*-inner product.*

Adaptive mesh-refining plays an important practical role in accurate calculation of the numerical solutions of partial differential equations, especially when the continuous solutions have local singularities or sharp layers. Adaptive mesh-refining algorithms consist of successive loops of SOLVE, ESTIMATE, MARK, and REFINE. A posteriori error estimators provide quantitative estimates for the actual error and motivate local mesh-refinement. Those are computed from the known values such as the given data of the problem and the computed numerical solutions. Various a posteriori error estimators for mixed finite element methods have been introduced and analyzed in [3,4] for the steady Stokes equations and in [5–7] for the Poisson equation. Only a few a posteriori error estimators for the mixed method for parabolic problems are available in the literature.

The mixed formulation computes simultaneously both the pressure and the flux, or displacements and stresses. The mixed finite element method has two important features; it conserves the mass locally and produces accurate flux even for highly nonhomogeneous media with large jumps in the physical properties. In many cases the mixed finite element method gives better approximations for the flux variable associated with the solution of a second order elliptic problem than the classical Galerkin method; see, for example, [2] for discussions.

In this paper, we present an a posteriori error estimates for the mixed formulation of linear parabolic problems. We consider the lowest order Raviart-Thomas finite elements for space discretization and a variant Crank-Nicolson scheme for time discretization; the variant Crank-Nicolson scheme is proposed so that mixed finite element spaces are permitted to change at different time levels. It is well known that the Crank-Nicolson type scheme has quadratic convergence. However, it was an open problem to get a posteriori error estimates of optimal order. Here, we obtain a posteriori error quantities of second order in time for the variant Crank-Nicolson mixed scheme. For this, we proceed in a similar way as in our early work [8] where continuous Galerkin finite element methods in space and a modified discontinous Galerkin scheme in time are considered. Optimal error estimates of second order in time were proved for the resulting scheme by exploiting the idea of the reconstruction introduced by Akrivis, Makridakis & Nochetto [1].

참고 문헌

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