KARCHER MEANS AND KARCHER EQUATIONS

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ABSTRACT

The Riemannian trace metric on the convex cone $\mathbb{P}$ of positive definite matrices of fixed size plays an important role in many applied areas involving matrix interpolation, filtering, estimation, convex programming, optimization and averaging, where it has been increasingly recognized that the Euclidean distance is often not the most suitable for the set $\mathbb{P}$ and that working with the appropriate geometry does matter in computational problems. It turns out that the Riemannian geometry plays a key role particularly in the study of "inversion" invariant data averaging procedures in image processing, in radar detection, in brain-computer interfacing and in subdivision schemes. The Karcher mean (or Cartan barycenter, least squares mean, Riemannian center of mass [1]) has recently become an important tool for the averaging and study of positive definite matrices. It is the unique minimizer of the sum of the squares of the Riemannian (trace) distances:

$$\Lambda(A_1, \ldots, A_n) = \underset{X \in \mathbb{P}}{\arg\min} \sum_{j=1}^{n} \delta^2(X, A_j)$$

and coincides with the unique positive definite solution of the *Karcher equation*

$$\sum_{j=1}^{n} \log(X^{\frac{1}{2}} A_j^{-1} X^{\frac{1}{2}}) = 0.$$

We present several methods to approximate the Karcher mean based on the strong law of large numbers [2], proximal point algorithm [7], and geometric power(ful) mean [3–6].

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